Continuity-Enhancing Degree Elevation and Splits for Splines

Supplemental Document

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S1 Parametric Continuity Formulations

Eq. 12 in the paper provides the solution for the modified control point $\mathbf{b}_k^*$ that minimizes the loss function

$$\mathbf{b}_k^* = \frac{1}{1+\alpha^{2k}}\mathbf{b}_k + \frac{\alpha^{2k}}{1+\alpha^{2k}}\mathbf{b}_{-k}^+ \ .$$

Expanding $\mathbf{b}_{-k}^+$ using Eq. 9 from the paper, we get update formulation for $\mathbf{b}_k^*$ in Eq. 13 in the paper, using

$$\begin{aligned}
\mathbf{b}_k^* &= \frac{1}{1+\alpha^{2k}}\mathbf{b}_k + \frac{\alpha^{2k}}{1+\alpha^{2k}}\left(\frac{1}{\alpha^k}\mathbf{D}_{-k} - \mathbf{D}_k + \mathbf{b}_k\right)\\
&= \frac{1}{1+\alpha^{2k}}\left(\mathbf{b}_k + \alpha^k\,\mathbf{D}_{-k} - \alpha^{2k}\,\mathbf{D}_k + \alpha^{2k}\,\mathbf{b}_k\right)\\
&= \mathbf{b}_k + \frac{1}{1+\alpha^{2k}}\left(\alpha^k\,\mathbf{D}_{-k} - \alpha^{2k}\,\mathbf{D}_k\right)\\
&= \mathbf{b}_k + \alpha^k\,\frac{\mathbf{D}_{-k} - \alpha^k\,\mathbf{D}_k}{1+\alpha^{2k}} \ .
\end{aligned}$$

To derive the update formula for $\mathbf{b}_{-k}^*$, we can start with Eq. 7 in the paper and regroup some terms

$$\mathbf{b}_{-k}^* = \mathbf{b}_{-k} + (-1)^k\alpha^k\left(\mathbf{D}_k - \frac{\mathbf{D}_{-k}}{\alpha^k} + \mathbf{b}_k^* - \mathbf{b}_k\right) \ .$$

Then, we expand $\mathbf{b}_k^*$ to derive the form in Eq. 13 in the paper

$$\begin{aligned}
\mathbf{b}_{-k}^* &= \mathbf{b}_{-k} + (-1)^k\alpha^k\left(\mathbf{D}_k - \frac{\mathbf{D}_{-k}}{\alpha^k} + \left(\cancel{\mathbf{b}_k} + \alpha^k\,\frac{\mathbf{D}_{-k} - \alpha^k\,\mathbf{D}_k}{1+\alpha^{2k}}\right) - \cancel{\mathbf{b}_k}\right)\\
&= \mathbf{b}_{-k} + (-1)^k\left(\alpha^k\,\mathbf{D}_k - \mathbf{D}_{-k} + \frac{\alpha^{2k}\mathbf{D}_{-k} - \alpha^{3k}\,\mathbf{D}_k}{1+\alpha^{2k}}\right)\\
&= \mathbf{b}_{-k} + (-1)^k\,\frac{(\alpha^k + \cancel{\alpha^{3k}})\mathbf{D}_k - (1 + \cancel{\alpha^{2k}})\mathbf{D}_{-k} + \cancel{\alpha^{2k}\mathbf{D}_{-k}} - \cancel{\alpha^{3k}\,\mathbf{D}_k}}{1+\alpha^{2k}}\\
&= \mathbf{b}_{-k} + (-1)^k\,\frac{\alpha^k\mathbf{D}_k - \mathbf{D}_{-k}}{1+\alpha^{2k}} \ .
\end{aligned}$$

S2 Eliminating the Singularity with Parametric Continuity Enhancement

Notice that there is a singularity with these equations when $\mathbf{b}_1 = \mathbf{b}_0$, resulting $\alpha = \infty$. To avoid this singularity, we can expand the terms with $\alpha = d_{-1}/d_1$, resulting

$$\Delta\mathbf{b}_{-k} = \frac{d_{-1}^k\,d_1^k\,\mathbf{D}_k - d_1^{2k}\,\mathbf{D}_{-k}}{d_1^{2k} + d_{-1}^{2k}} \ . \tag{S.1}$$

$$\Delta\mathbf{b}_{k} = \frac{d_{-1}^k\,d_1^k\,\mathbf{D}_{-k} - d_{-1}^{2k}\,\mathbf{D}_{k}}{d_1^{2k} + d_{-1}^{2k}} \tag{S.2}$$

Here, there is a singularity only when both $\mathbf{b}_1 = \mathbf{b}_{-1} = \mathbf{b}_0$, in which case there is a trivial solution of $\mathbf{b}_k = \mathbf{b}_{-k} = \mathbf{b}_0$ for all k. Note that when either $\mathbf{b}_1 = \mathbf{b}_0$ or $\mathbf{b}_{-1} = \mathbf{b}_0$ (but not both), the curve is not C^1 continuous; therefore, attempting to achieve C^k continuity is futile. Still, Equations S.2 and S.1 offer numerical stability when $\mathbf{b}_1$ or $\mathbf{b}_{-1}$ is close to $\mathbf{b}_0$ and that the result is consistent when either are at $\mathbf{b}_0$ or when they both approach to $\mathbf{b}_0$.

S3 Geometric Continuity Conditions

The geometric continuity conditions Eq. 15-18 in the paper up to G^4 can be written as

$$\mathbf{D}_{-1} = \alpha\,\mathbf{D}_1 \tag{S.3}$$

$$\mathbf{D}_{-2} = \alpha^2\,\mathbf{D}_2 + \frac{\beta\,\mathbf{D}_1}{n-1} \tag{S.4}$$

$$\mathbf{D}_{-3} = \alpha^3\,\mathbf{D}_3 + \frac{3\alpha\beta\,\mathbf{D}_2}{n-2} + \frac{\gamma\,\mathbf{D}_1}{(n-1)(n-2)} \tag{S.5}$$

$$\mathbf{D}_{-4} = \alpha^4\,\mathbf{D}_4 + \frac{6\alpha^2\beta\,\mathbf{D}_3}{n-3} + \frac{(4\alpha\gamma + 3\beta^2)\mathbf{D}_2}{(n-2)(n-3)} + \frac{(n-4)!}{(n-1)!}\delta\,\mathbf{D}_1 \tag{S.6}$$

S3.1 Derivative Identities

For simplifying the equations, we use the identities

$$\mathbf{D}_{-2} = \mathbf{D}_{-1} + (\mathbf{b}_{-2} - \mathbf{b}_{-1}) \tag{S.7}$$

$$\mathbf{D}_{-3} = 2\mathbf{D}_{-2} - \mathbf{D}_{-1} - (\mathbf{b}_{-3} - \mathbf{b}_{-2}) \tag{S.8}$$

$$\mathbf{D}_{-4} = 3\mathbf{D}_{-3} - 3\mathbf{D}_{-2} + \mathbf{D}_{-1} + (\mathbf{b}_{-4} - \mathbf{b}_{-3}) \tag{S.9}$$

which come from the definition of $\mathbf{D}_{-k}$ in Eq. 3 in the paper, and similarly

$$\mathbf{D}_2 = -\mathbf{D}_1 + (\mathbf{b}_2 - \mathbf{b}_1) \tag{S.10}$$

$$\mathbf{D}_3 = -2\mathbf{D}_2 - \mathbf{D}_1 + (\mathbf{b}_3 - \mathbf{b}_2) \tag{S.11}$$

$$\mathbf{D}_4 = -3\mathbf{D}_3 - 3\mathbf{D}_2 - \mathbf{D}_1 + (\mathbf{b}_4 - \mathbf{b}_3) \tag{S.12}$$

which come from the definition of $\mathbf{D}_k$ in Eq. 3 in the paper.

S3.2 G^2 Continuity

To derive the simplified rule for G^2 continuity, we first insert Eq. S.3 into Eq. S.7 to get

$$\mathbf{D}_{-2} = \alpha\,\mathbf{D}_1 + (\mathbf{b}_{-2} - \mathbf{b}_{-1}) \ .$$

Then, we expand $\mathbf{D}_{-2}$ using Eq. S.4

$$\alpha\,\mathbf{D}_1 + (\mathbf{b}_{-2} - \mathbf{b}_{-1}) = \alpha^2\,\mathbf{D}_2 + \frac{\beta\,\mathbf{D}_1}{n-1}$$

$$\mathbf{b}_{-2} - \mathbf{b}_{-1} = \alpha^2\,\mathbf{D}_2 + \left(\frac{\beta}{n-1} - \alpha\right)\mathbf{D}_1 \tag{S.13}$$

and finally expand $\mathbf{D}_2$ using Eq. S.10, matching Eq. 19 in the paper with modified positions for $\mathbf{b}_{-2}$ and $\mathbf{b}_2$

$$\mathbf{b}_{-2} - \mathbf{b}_{-1} = \alpha^2(\mathbf{b}_2 - \mathbf{b}_1) + \left(\frac{\beta}{n-1} - \alpha^2 - \alpha\right)\mathbf{D}_1 \ . \tag{S.14}$$

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S3.3 G^3 Continuity

The simplified rule for G^3 can be found similarly, starting with Eq. S.8 and expanding $\mathbf{D}_{-1}$, $\mathbf{D}_{-2}$, and $\mathbf{D}_{-2}$, using the G^1, G^2, and G^3 rules in Eq. S.3 through S.5, resulting

$$\alpha^3\,\mathbf{D}_3 + \frac{3\alpha\beta\,\mathbf{D}_2}{n-2} + \frac{\gamma\,\mathbf{D}_1}{(n-1)(n-2)} = 2\alpha^2\,\mathbf{D}_2 + \left(\frac{2\beta}{n-1} - \alpha\right)\mathbf{D}_1 - (\mathbf{b}_{-3} - \mathbf{b}_{-2})$$

Then, we expand $\mathbf{D}_3$ using Eq. S.11 to get

$$\alpha^3\Big(-2\mathbf{D}_2 - \mathbf{D}_1 + (\mathbf{b}_3 - \mathbf{b}_2)\Big) + \frac{3\alpha\beta\,\mathbf{D}_2}{n-2} + \frac{\gamma\,\mathbf{D}_1}{(n-1)(n-2)} = 2\alpha^2\,\mathbf{D}_2 + \left(\frac{2\beta}{n-1} - \alpha\right)\mathbf{D}_1 - (\mathbf{b}_{-3} - \mathbf{b}_{-2})$$

Rearranging the terms, we get a variation of Eq. 25 in the paper with modified positions for $\mathbf{b}_{-3}$ and $\mathbf{b}_3$

$$\mathbf{b}_{-3} - \mathbf{b}_{-2} = \left(-\frac{3\alpha\beta}{n-2} + 2\alpha^3 + 2\alpha^2\right)\mathbf{D}_2 + \left(-\frac{\gamma}{(n-1)(n-2)} + \frac{2\beta}{n-1} + \alpha^3 - \alpha\right)\mathbf{D}_1 - \alpha^3(\mathbf{b}_3 - \mathbf{b}_2)$$

S3.4 G^4 Continuity

We derive the simplified rule for G^4 similarly, starting with Eq. S.9 and expanding $\mathbf{D}_{-1}$, $\mathbf{D}_{-2}$, $\mathbf{D}_{-3}$, and $\mathbf{D}_{-4}$, using the G^1, G^2, G^3, and G^4 rules in Eq. S.3 through S.6, resulting

$$\alpha^4\,\mathbf{D}_4 + \frac{6\alpha^2\beta\,\mathbf{D}_3}{n-3} + \frac{(4\alpha\gamma + 3\beta^2)\mathbf{D}_2}{(n-2)(n-3)} + \frac{\delta\,\mathbf{D}_1}{(n-1)(n-2)(n-3)} = 3\left(\alpha^3\,\mathbf{D}_3 + \frac{3\alpha\beta\,\mathbf{D}_2}{n-2} + \frac{\gamma\,\mathbf{D}_1}{(n-1)(n-2)}\right) - 3\left(\alpha^2\,\mathbf{D}_2 + \frac{\beta\,\mathbf{D}_1}{n-1}\right) + \alpha\mathbf{D}_1 + (\mathbf{b}_{-4} - \mathbf{b}_{-3})\ .$$

Combining all terms with $\mathbf{D}_1$, $\mathbf{D}_2$, and $\mathbf{D}_3$, we get

$$\mathbf{b}_{-4} - \mathbf{b}_{-3} = \alpha^4\,\mathbf{D}_4 + \left(\frac{6\alpha^2\beta}{n-3} - 3\alpha^3\right)\mathbf{D}_3 + \left(\frac{4\alpha\gamma + 3\beta^2}{(n-2)(n-3)} - \frac{9\alpha\beta}{n-2} + 3\alpha^2\right)\mathbf{D}_2 + \left(\frac{\delta}{(n-1)(n-2)(n-3)} - \frac{3\gamma}{(n-1)(n-2)} + \frac{3\beta}{n-1} - \alpha\right)\mathbf{D}_1\ .$$

Finally, we expand $\mathbf{D}_4$ using Eq. S.12 to get Eq. 32 in the paper

$$\mathbf{b}_{-4} - \mathbf{b}_{-3} = \alpha^4(\mathbf{b}_4 - \mathbf{b}_3) + \left(\frac{6\alpha^2\beta}{n-3} - 3\alpha^3 - 3\alpha^4\right)\mathbf{D}_3 + \left(\frac{4\alpha\gamma + 3\beta^2}{(n-2)(n-3)} - \frac{9\alpha\beta}{n-2} + 3\alpha^2 - 3\alpha^4\right)\mathbf{D}_2 + \left(\frac{\delta}{(n-1)(n-2)(n-3)} - \frac{3\gamma}{(n-1)(n-2)} + \frac{3\beta}{n-1} - \alpha - \alpha^4\right)\mathbf{D}_1\ .$$

S4 Degree Elevation for Bézier Curves

Given a Bézier curve $\mathbf{B}(t)$ of degree n with control points $\mathbf{b}'_i$, where $i \in \{0, 1, \cdots, n\}$, we can convert it into a curve with degree $n+1$ without modifying the shape or the parameterization of the curve, using

$$\mathbf{B}(t) = (1-t)\,\mathbf{B}(t) + t\,\mathbf{B}(t)\ . \tag{S.15}$$

The resulting polynomial can be written with a new set of Bézier control points $\mathbf{b}'_i$, where $i \in \{0, 1, \cdots, n+1\}$, such that

$$\mathbf{B}(t) = \sum_{i=0}^{n+1} \binom{n+1}{i} (1-t)^{n+1-i}\, t^i\, \mathbf{b}'_i\ , \tag{S.16}$$

where the new control points can be calculated as

$$\mathbf{b}'_i = \frac{i}{n+1}\,\mathbf{b}_{i-1} + \left(1 - \frac{i}{n+1}\right)\mathbf{b}_i\ . \tag{S.17}$$

When the input degree n is odd, we can apply two degree elevations in one step, using

$$\mathbf{B}(t) = (1-t)^2\mathbf{B}(t) + 2(1-t)t\mathbf{B}(t) + t^2\mathbf{B}(t)\ . \tag{S.18}$$

Again, the resulting polynomial can be written with a new set of Bézier control points $\mathbf{b}''_i$, where $i \in \{0, 1, \cdots, n+2\}$, such that

$$\mathbf{B}(t) = \sum_{i=0}^{n+2} \binom{n+2}{i} (1-t)^{n+2-i}\, t^i\, \mathbf{b}''_i\ , \tag{S.19}$$

where the new control points can be calculated as

$$\mathbf{b}''_i = \frac{i(i-1)}{(n+1)(n+2)}\,\mathbf{b}_{i-2} + \frac{2i}{n+1}\left(1 - \frac{i}{n+2}\right)\mathbf{b}_{i-1} + \left(1 - \frac{i}{n+1}\right)\left(1 - \frac{i}{n+2}\right)\mathbf{b}_i\ . \tag{S.20}$$

Note that, while these formulations include terms $\mathbf{b}_i$ with $i < 0$ and $i > n$, they can be omitted, as their multipliers are zero.

In the specific case of piecewise cubic ($n = 3$) splines with C^1 continuity, which are commonplace, we elevate them to quintic ($n' = 5$) polynomials, using

$$\mathbf{b}'_i = \frac{i(i-1)}{20}\,\mathbf{b}_{i-2} + \frac{i(5-i)}{10}\,\mathbf{b}_{i-1} + \frac{(4-i)(5-i)}{20}\,\mathbf{b}_i\ . \tag{S.21}$$

S5 Derivation of Continuity-Enhancing Splits

For our continuity-enhancing splits, we generate the control points $\mathbf{p}_i$ and $\mathbf{r}_i$ for the first and the last pieces following Bézier subdivision formulas at normalized parameters λ and $1-\lambda$. The control points $\mathbf{q}_i$ of the middle piece are determined following C^2 continuity constraints.

The C^1 continuity equations can be written as

$$(1-2\lambda)(\mathbf{q}_0 - \mathbf{p}_2) = \lambda(\mathbf{q}_1 - \mathbf{q}_0) \tag{S.22}$$

$$(1-2\lambda)(\mathbf{q}_3 - \mathbf{r}_1) = \lambda(\mathbf{q}_2 - \mathbf{q}_3) \tag{S.23}$$

Similarly, the C^2 continuity equations are

$$(1-2\lambda)^2(\mathbf{p}_1 - 2\mathbf{p}_2) = (\lambda^2 - (1-2\lambda)^2)\mathbf{q}_0 + \lambda^2\mathbf{q}_2 - 2\lambda^2\mathbf{q}_1 \tag{S.24}$$

$$(1-2\lambda)^2(\mathbf{r}_2 - 2\mathbf{r}_1) = (\lambda^2 - (1-2\lambda)^2)\mathbf{q}_3 + \lambda^2\mathbf{q}_1 - 2\lambda^2\mathbf{q}_2 \tag{S.25}$$

From C^1 continuity equations above, we can write

$$\lambda \mathbf{q}_1 = (1-\lambda)\mathbf{q}_0 - (1-2\lambda)\mathbf{p}_2 \tag{S.26}$$

$$\lambda \mathbf{q}_2 = (1-\lambda)\mathbf{q}_3 - (1-2\lambda)\mathbf{r}_1 \tag{S.27}$$

Substituting these into the C^2 continuity equations, we get

$$\begin{aligned}\lambda(1-2\lambda)^2(\mathbf{p}_1 - 2\mathbf{p}_2) &= \lambda(\lambda^2 - (1-2\lambda)^2)\mathbf{q}_0 \\ &\quad + \lambda^2\Big((1-\lambda)\mathbf{q}_3 - (1-2\lambda)\mathbf{r}_1\Big) - 2\lambda^2\Big((1-\lambda)\mathbf{q}_0 - (1-2\lambda)\mathbf{p}_2\Big) \\ \lambda(1-2\lambda)^2(\mathbf{r}_2 - 2\mathbf{r}_1) &= \lambda(\lambda^2 - (1-2\lambda)^2)\mathbf{q}_3 \\ &\quad + \lambda^2\Big((1-\lambda)\mathbf{q}_0 - (1-2\lambda)\mathbf{p}_2\Big) - 2\lambda^2\Big((1-\lambda)\mathbf{q}_3 - (1-2\lambda)\mathbf{r}_1\Big) .\end{aligned}$$

Moving $\mathbf{q}_0$ and $\mathbf{q}_3$ to one side and simplifying the terms with λ forms

$$\begin{aligned}&\lambda(1-2\lambda)^2\mathbf{p}_1 - 2\lambda(1-2\lambda)(1-\lambda)\mathbf{p}_2 + \lambda^2(1-2\lambda)\mathbf{r}_1 \\ &\qquad = \lambda^2(1-\lambda)\mathbf{q}_3 - \lambda(1-\lambda)^2\mathbf{q}_0 \\ &\lambda(1-2\lambda)^2\mathbf{r}_2 - 2\lambda(1-2\lambda)(1-\lambda)\mathbf{r}_1 + \lambda^2(1-2\lambda)\mathbf{p}_2 \\ &\qquad = \lambda^2(1-\lambda)\mathbf{q}_0 - \lambda(1-\lambda)^2\mathbf{q}_3\end{aligned}$$

We can combine them to get $\mathbf{q}_0$ by first multiplying the equations by $(1-\lambda)$ and λ, respectively

$$\begin{aligned}&\lambda(1-2\lambda)^2(1-\lambda)\mathbf{p}_1 - 2\lambda(1-2\lambda)(1-\lambda)^2\mathbf{p}_2 \\ &\qquad + \lambda^2(1-2\lambda)(1-\lambda)\mathbf{r}_1 = \lambda^2(1-\lambda)^2\mathbf{q}_3 - \lambda(1-\lambda)^3\mathbf{q}_0 \\ &\lambda^2(1-2\lambda)^2\mathbf{r}_2 - 2\lambda^2(1-2\lambda)(1-\lambda)\mathbf{r}_1 \\ &\qquad + \lambda^3(1-2\lambda)\mathbf{p}_2 = \lambda^3(1-\lambda)\mathbf{q}_0 - \lambda^2(1-\lambda)^2\mathbf{q}_3\end{aligned}$$

and then adding them together and simplifying the terms, resulting

$$\begin{aligned}-(1-\lambda)\mathbf{q}_0 &= (1-2\lambda)(1-\lambda)\mathbf{p}_1 - \Big((1-\lambda)^2 + (1-2\lambda)\Big)\mathbf{p}_2 \\ &\quad - \lambda(1-\lambda)\mathbf{r}_1 + \lambda(1-2\lambda)\mathbf{r}_2\end{aligned} \tag{S.28}$$

Similarly, we can first multiply the equations by λ and $(1-\lambda)$, respectively

$$\begin{aligned}&\lambda^2(1-2\lambda)^2\mathbf{p}_1 - 2\lambda^2(1-2\lambda)(1-\lambda)\mathbf{p}_2 \\ &\qquad + \lambda^3(1-2\lambda)\mathbf{r}_1 = \lambda^3(1-\lambda)\mathbf{q}_3 - \lambda^2(1-\lambda)^2\mathbf{q}_0 \\ &\lambda(1-2\lambda)^2(1-\lambda)\mathbf{r}_2 - 2\lambda(1-2\lambda)(1-\lambda)^2\mathbf{r}_1 \\ &\qquad + \lambda^2(1-2\lambda)(1-\lambda)\mathbf{p}_2 = \lambda^2(1-\lambda)^2\mathbf{q}_0 - \lambda(1-\lambda)^3\mathbf{q}_3\end{aligned}$$

and then adding them together and simplifying the terms, we get

$$\begin{aligned}-(1-\lambda)\mathbf{q}_3 &= (1-2\lambda)(1-\lambda)\mathbf{r}_2 - \Big((1-\lambda)^2 + (1-2\lambda)\Big)\mathbf{r}_1 \\ &\quad - \lambda(1-\lambda)\mathbf{p}_2 + \lambda(1-2\lambda)\mathbf{p}_1\end{aligned} \tag{S.29}$$

The formulas for $\mathbf{q}_0$ and $\mathbf{q}_3$ come from Equations S.28 and S.29. Plugging them into Equations S.26 and S.27, we get the formulas for $\mathbf{q}_1$ and $\mathbf{q}_2$.

The formulas for $\mathbf{q}_i$ with $\lambda = \frac{1}{4}$ are

$$\mathbf{q}_0 = \frac{17}{12}\mathbf{p}_2 - \frac{1}{2}\mathbf{p}_1 + \frac{1}{4}\mathbf{r}_1 - \frac{1}{6}\mathbf{r}_2 \tag{S.30}$$

$$\mathbf{q}_1 = 3\mathbf{q}_0 - 2\mathbf{p}_2 \tag{S.31}$$

$$\mathbf{q}_2 = 3\mathbf{q}_3 - 2\mathbf{r}_1 \tag{S.32}$$

$$\mathbf{q}_3 = \frac{17}{12}\mathbf{r}_1 - \frac{1}{2}\mathbf{r}_2 + \frac{1}{4}\mathbf{p}_2 - \frac{1}{6}\mathbf{p}_1 . \tag{S.33}$$

or in matrix form

$$\begin{bmatrix}\mathbf{q}_0\\ \mathbf{q}_1\\ \mathbf{q}_2\\ \mathbf{q}_3\end{bmatrix} = \frac{1}{12}\begin{bmatrix}-6 & 17 & 3 & -2\\ -18 & 27 & 9 & -6\\ -6 & 9 & 27 & -18\\ -2 & 3 & 17 & -6\end{bmatrix}\begin{bmatrix}\mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{r}_1\\ \mathbf{r}_2\end{bmatrix} \tag{S.34}$$

where

$$\mathbf{p}_0 = \mathbf{b}_0 \tag{S.35}$$

$$\mathbf{p}_1 = (3/4)\mathbf{b}_0 + (1/4)\mathbf{b}_1 \tag{S.36}$$

$$\mathbf{p}_2 = (9/16)\mathbf{b}_0 + (6/16)\mathbf{b}_1 + (1/16)\mathbf{b}_2^* \tag{S.37}$$

$$\mathbf{p}_3 = \mathbf{q}_0 \tag{S.38}$$

$$\mathbf{r}_0 = \mathbf{q}_3 \tag{S.39}$$

$$\mathbf{r}_1 = (1/16)\,\mathbf{b}_1^* + (6/16)\mathbf{b}_2 + (9/16)\mathbf{b}_3 \tag{S.40}$$

$$\mathbf{r}_2 = (1/4)\mathbf{b}_2 + (3/4)\mathbf{b}_3 \tag{S.41}$$

$$\mathbf{r}_3 = \mathbf{b}_3 . \tag{S.42}$$

All formulas for $\mathbf{p}_i$, $\mathbf{q}_i$, and $\mathbf{r}_i$ with $\lambda = \frac{1}{4}$ for continuity-enhancing splits can be written in a matrix form as

$$\begin{bmatrix}\mathbf{p}_0\\ \mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{q}_0\\ \mathbf{q}_1\\ \mathbf{q}_2\\ \mathbf{q}_3\\ \mathbf{r}_1\\ \mathbf{r}_2\\ \mathbf{r}_3\end{bmatrix} = \frac{1}{192}\begin{bmatrix}192 & 0 & 0 & 0 & 0 & 0\\ 144 & 48 & 0 & 0 & 0 & 0\\ 108 & 72 & 12 & 0 & 0 & 0\\ 81 & 78 & 17 & 3 & 10 & 3\\ 27 & 90 & 27 & 9 & 30 & 9\\ 9 & 30 & 9 & 27 & 90 & 27\\ 3 & 10 & 3 & 17 & 78 & 81\\ 0 & 0 & 0 & 12 & 72 & 108\\ 0 & 0 & 0 & 0 & 48 & 144\\ 0 & 0 & 0 & 0 & 0 & 192\end{bmatrix}\begin{bmatrix}\mathbf{b}_0\\ \mathbf{b}_1\\ \mathbf{b}_2^*\\ \mathbf{b}_1^*\\ \mathbf{b}_2\\ \mathbf{b}_3\end{bmatrix} \tag{S.43}$$

S6 Optimization with Linearity Constraint

The linearity constraint for solving the optimization problem to achieve C^k continuity

$$\mathbf{b}_k^* = \mathbf{b}_0 + u_k\hat{\mathbf{d}} , \qquad \text{where} \qquad \hat{\mathbf{d}} = \frac{\mathbf{b}_1 - \mathbf{b}_{-1}}{\|\mathbf{b}_1 - \mathbf{b}_{-1}\|} ,$$

changes the loss function to

$$\mathcal{L} = \|\mathbf{b}_0 + u_k\hat{\mathbf{d}} - \mathbf{b}_k\|^2 + \alpha^{2k}\|\mathbf{b}_0 + u_k\hat{\mathbf{d}} - \mathbf{b}_{-k}^+\|^2 .$$

We can find u_k that minimizes the loss function by taking the derivative of the loss function with respect to u_k and setting it to zero, resulting

$$0 = \frac{\partial \mathcal{L}}{\partial u_k} = 2\hat{\mathbf{d}} \cdot \Big(\big(\mathbf{b}_0 + u_k\hat{\mathbf{d}} - \mathbf{b}_k\big) + \alpha^{2k}\big(\mathbf{b}_0 + u_k\hat{\mathbf{d}} - \mathbf{b}_{-k}^+\big)\Big) .$$

Removing the factor of 2 and combining the terms with u_k, we get

$$0 = u_k(1+\alpha^{2k}) + \hat{\mathbf{d}} \cdot \Big(\big(\mathbf{b}_0 - \mathbf{b}_k\big) + \alpha^{2k}\big(\mathbf{b}_0 - \mathbf{b}_{-k}^+\big)\Big)$$

Rearranging the terms, we get a closed-form solution for u_k

$$u_k = \hat{\mathbf{d}} \cdot \frac{\big(\mathbf{b}_k - \mathbf{b}_0\big) + \alpha^{2k}\big(\mathbf{b}_{-k}^+ - \mathbf{b}_0\big)}{1+\alpha^{2k}} = \hat{\mathbf{d}} \cdot \left(\frac{\mathbf{b}_k + \alpha^{2k}\mathbf{b}_{-k}^+}{1+\alpha^{2k}} - \mathbf{b}_0\right) .$$

Expanding $\mathbf{b}^+_{-k}$ using Eq. 9, we get Eq. 56 in the paper, such that

$$\begin{aligned} u_k &= \hat{\mathbf{d}} \cdot \left(\frac{\alpha^k \mathbf{D}_{-k} - \alpha^{2k} \mathbf{D}_k + (1+\alpha^{2k}) \mathbf{b}_k}{1+\alpha^{2k}} - \mathbf{b}_0 \right) \\ &= \hat{\mathbf{d}} \cdot \left(\alpha^k \frac{\mathbf{D}_{-k} - \alpha^k \mathbf{D}_k}{1+\alpha^{2k}} + \mathbf{b}_k - \mathbf{b}_0 \right) = \hat{\mathbf{d}} \cdot (\Delta \mathbf{b}_k + \mathbf{b}_k - \mathbf{b}_0) \ . \end{aligned}$$

Using this in Eq. 54 from the paper, we can find $\mathbf{b}^*_k$. Then, we can evaluate the other control point on the other side using Eq. 7 from the paper, such that

$$\mathbf{b}^*_{-k} = (-1)^k \, \alpha^k \left(\mathbf{D}_k + \mathbf{b}^*_k - \mathbf{b}_k \right) - (-1)^k \, \mathbf{D}_{-k} + \mathbf{b}_{-k} \ .$$

Here, we can expand $\mathbf{b}^*_k$ with a similar linear constraint

$$\mathbf{b}^*_{-k} = \mathbf{b}_0 + u_{-k} \hat{\mathbf{d}} \ ,$$

and expand $\mathbf{b}^*_k$ to get

$$u_{-k} \hat{\mathbf{d}} = (-1)^k \, \alpha^k \left(\mathbf{D}_k + \mathbf{b}_0 + u_k \hat{\mathbf{d}} - \mathbf{b}_k \right) - (-1)^k \, \mathbf{D}_{-k} + \mathbf{b}_{-k} - \mathbf{b}_0$$

Applying a dot product with $\hat{\mathbf{d}}$ for both sides forms

$$\begin{aligned} u_{-k} &= \hat{\mathbf{d}} \cdot \left((-1)^k \alpha^k (\mathbf{D}_k - \mathbf{b}_k + \mathbf{b}_0) - (-1)^k \, \mathbf{D}_{-k} + \mathbf{b}_{-k} - \mathbf{b}_0 \right) \\ &\qquad + (-1)^k \, \alpha^k u_k \ . \end{aligned}$$

Expanding u_k and simplifying terms, we get

$$\begin{aligned} u_{-k} &= \hat{\mathbf{d}} \cdot \Big((-1)^k \left(\alpha^k (\mathbf{D}_k \cancel{- \mathbf{b}_k + \mathbf{b}_0}) - \mathbf{D}_{-k} + \alpha^k (\Delta \mathbf{b}_k \cancel{+ \mathbf{b}_k - \mathbf{b}_0}) \right) \\ &\qquad + \mathbf{b}_{-k} - \mathbf{b}_0 \Big) \end{aligned}$$

Finally, expanding $\Delta \mathbf{b}_k$, we get Eq. 58 in the paper

$$\begin{aligned} u_{-k} &= \hat{\mathbf{d}} \cdot \left((-1)^k \left(\alpha^k \mathbf{D}_k - \mathbf{D}_{-k} + \alpha^{2k} \frac{\mathbf{D}_{-k} - \alpha^k \mathbf{D}_k}{1+\alpha^{2k}} \right) + \mathbf{b}_{-k} - \mathbf{b}_0 \right) \\ &= \hat{\mathbf{d}} \cdot \left((-1)^k \frac{\alpha^k \mathbf{D}_k - \mathbf{D}_{-k}}{1+\alpha^{2k}} + \mathbf{b}_{-k} - \mathbf{b}_0 \right) \\ &= \hat{\mathbf{d}} \cdot \left((-1)^k \Delta \mathbf{b}_{-k} + \mathbf{b}_{-k} - \mathbf{b}_0 \right) \ . \end{aligned}$$

S7 Convex Hull Property

If desired, the convex hull property can always be maintained with continuity enhancement.

Proof. To prove this, it is sufficient to show that there exists at least one pair of modified control points $\mathbf{b}^*_k$ and $\mathbf{b}^*_{-k}$ around an interpolated control point $\mathbf{b}_0$ that satisfies the C^k continuity condition while both $\mathbf{b}^*_k$ and $\mathbf{b}^*_{-k}$ are within the convex hulls of the two neighboring pieces joining at $\mathbf{b}_0$. One solution is using $\mathbf{b}^*_k = \mathbf{b}_{k-1}$ and $\mathbf{b}^*_{-k} = \mathbf{b}_{-k+1}$. This obviously maintains the convex hull property. To show that it achieves C^k continuity, we can insert them into Eq. 6 in the paper, resulting

$$\mathbf{D}_{-k} + (-1)^k (\mathbf{b}_{-k+1} - \mathbf{b}_{-k}) = \alpha^k \left(\mathbf{D}_k + \mathbf{b}_{k-1} - \mathbf{b}_k \right) \ .$$

From the definition of $\mathbf{D}_k$ and $\mathbf{D}_{-k}$, we write them as

$$\begin{aligned} \mathbf{b}_{-k} - \mathbf{b}_{-k+1} &= \sum_{i=1}^{k} (-1)^{i+k} \binom{k-1}{i-1} \mathbf{D}_{-i} \\ \mathbf{b}_k - \mathbf{b}_{k-1} &= \sum_{i=1}^{k} (-1)^{i+1} \binom{k-1}{i-1} \mathbf{D}_i \ . \end{aligned}$$

Inserting them into the equation above, the $\mathbf{D}_k$ and $\mathbf{D}_{-k}$ terms cancel out, resulting

$$\sum_{i=1}^{k-1} (-1)^i \binom{k-1}{i-1} \mathbf{D}_{-i} = \alpha^k \sum_{i=1}^{k-1} (-1)^{i+1} \binom{k-1}{i-1} \mathbf{D}_i \ .$$

Since the input curve to continuity enhancement must support C^{k-1} continuity, each term of the sum on either side match, satisfying the condition.